Another characteristic evident in Fig. 3 is that the low supersonic contours ($M=1.2$) are generally more convergent than are the high subsonic ones ($M=0.8$). This figure demonstrates the ability of the series solution to provide initial value data for space-marching method of characteristics or finite difference analyses of radial supersonic nozzles.

Acknowledgment

This research was supported in part by an Office of Naval Research Graduate Fellowship for B. F. Carroll.

References

[1]Conley, R. R., Hoffman, J. D., and Thompson, H. D., "An Analytical and Experimental Investigation of Annular Propulsive Nozzles," AIAA Paper 84-0282, Jan. 1984.

[2]Dutton, J. C. and Addy, A. L., "Transonic Flow in the Throat Region of Axisymmetric Nozzles," *AIAA Journal*, Vol. 19, June 1981, pp. 801-804.

[3]Dutton, J. C. and Addy, A. L., "Transonic Flow in the Throat Region of Annular Supersonic Nozzles," *AIAA Journal*, Vol. 20, Sept. 1982, pp. 1236-1243.

[4]Carroll, B. F. and Dutton, J. C., "Transonic Throat Flow in Radial or Nearly Radial Supersonic Nozzles," Texas A&M University, College Station, TEES Rept. FP-84-01, May 1984.

[5]Hall, I. M., "Transonic Flow in Two-Dimensional and Axially-Symmetric Nozzles," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. XV, Pt. 4, 1962, pp. 487-508.

[6]Carroll, B. F. and Dutton, J. C., "RATLE, A Computer Program for Radial Transonic Nozzles," Texas A&M University, College Station, TEES Rept. FP-84-91, June 1984.

On the Vortex Stretching Modification of the k-ϵ Turbulence Model: Radial Jets

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Introduction

ALTHOUGH the k-ϵ turbulence model has had success predicting free shear flows, it and other two-equation models appear to require a different set of constants to match both plane and round jet growth rates.[1-4] For example, Launder et al.[1] modified the eddy-viscosity coefficient C_μ and the destruction of dissipation constant $C_{\epsilon 2}$ to achieve round jet agreement. In the far field this modification reduced to a rather large C_μ correction. Raiszadeh and Dwyer[5] showed that k-ϵ model results are quite sensitive to the dissipation equation model constants $C_{\epsilon 1}$ and $C_{\epsilon 2}$; therefore, it is not surprising that these have been the target of several turbulence modelers.[4,6]

Pope[4] has given a phenomenological argument for incorporating a vortex stretching invariant term in the dissipation transport equation to modify the source terms. Unlike other models,[1,6] the form of the resulting equation has general application to three-dimensional problems. Pope settled on a third constant, $C_{\epsilon 3}$, such that his results matched round jet growth rate data. Plane jet agreement is guaranteed since there is no vortex stretching.

Performance of this model on the calculation of radially spreading jets has not been examined, although this case provides an excellent test[7] of the model. The radial jet is axisymmetric, like the round jet, but its velocity decays in the manner of a plane jet. In this Note the invariant vortex stretching modification to the k-ϵ model is applied to the self-preserving radial jet and shown to be inadequate.

Received June 25, 1984.

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Analysis

The present study of free jets in stagnant surroundings, using a k-ϵ model, is based on a far-field similarity formulation.[8,9] Similarity variables may be defined by

$$\frac{u}{u_0}=U(\eta),\quad \frac{k}{u_0^2}=G(\eta),\quad \frac{\epsilon x}{u_0^3}=H(\eta),\quad \frac{v}{u_0}=V(\eta),$$

$$\eta=\frac{y}{C_\mu^{1/2}x} \tag{1}$$

$$\eta^m U=(\eta^m F)',\quad C_\mu^{-1/2}V=\eta U-2^{j-1}F \tag{2}$$

where u,v and x,y are the streamwise, transverse velocities and coordinates, respectively. At $y=0$, the velocity $u_0(x)$ follows an $x^{-(j+1)/2}$ behavior. For plane jets, $j=m=0$; for radial jets, $j=1$, $m=0$, and for round jets, $j=m=1$. Primes indicate differentiation with respect to the similarity variable, η. Use of the stream function $\eta^m F$ in Eqs. (1) and (2) ensures the conservation of mass; the remaining thin shear layer equations become

Momentum:

$$2^{j-1}FU+\frac{G^2}{H}U'=0 \tag{3}$$

Energy:

$$2^jUG+2^{j-1}FG'+\frac{\sigma_k^{-1}}{\eta^m}\left(\eta^m\frac{G^2}{H}G'\right)'+\frac{G^2}{H}U'^2-H=0 \tag{4}$$

Dissipation:

$$\frac{1}{2}(5+3j)UH+2^{j-1}FH'+\frac{\sigma_\epsilon^{-1}}{\eta^m}\left(\eta^m\frac{G^2}{H}H'\right)'+C_{\epsilon 1}GU'^2$$
$$-C_{\epsilon 2}\frac{H^2}{G}+\underline{j\bar{C}_{\epsilon 3}\frac{G^2}{H}U'^2F'}=0 \tag{5}$$

where the underlined term of Eq. (5) represents the vortex stretching modification of Pope. Model constants are given by

$$C_\mu=0.09,\quad \sigma_k=1,\quad \sigma_\epsilon=1.3,$$
$$C_{\epsilon 1}=1.44,\quad C_{\epsilon 2}=1.90,\quad \bar{C}_{\epsilon 3}=C_{\epsilon 3}/4C_\mu=2.194 \tag{6}$$

Boundary conditions for the system are

$$\eta=0:\quad F=0,\quad U=1,\quad G'=0,\quad H'=0 \tag{7a}$$

$$\eta=\eta_e:\quad G=0,\quad H=0 \tag{7b}$$

Table 1 Comparison of calculated and observed jet growth rates

		$y_{1/2}/x$	
Jet	k-ϵ model	Calculated	Data[3,10,11]
Plane	—	0.1080	0.10-0.11
Round	Original	0.1199	
	Modified	0.0858	0.086
Radial	Original	0.0951	0.096-0.11
	Modified	0.0400	

where subscript e indicates the edge of the shear layer. It is immediately apparent from Eqs. (3-5) that the solution is independent of C_μ so that the Launder et al. correction of C_μ is directly translated to a growth rate modification via $y_{1/2}/x = C_\mu^{1/2}\eta_{1/2}$.

The form of the vortex stretching term of Eq. (5) is derived from Pope's equations for axisymmetric flow.[4] That is,

$$C_{\epsilon 3}\chi = \frac{C_{\epsilon 3}}{4}\left(\frac{k}{\epsilon}\right)^3\left(\frac{\partial q_z}{\partial r} - \frac{\partial q_r}{\partial z}\right)^2\frac{q_r}{r} \tag{8a}$$

where χ is the invariant, r the radial direction, z the direction of the axis of symmetry and q_r and q_z are the respective velocities. Pope requires $C_{\epsilon 3} = 0.79$ to reproduce the round jet growth rate. Substituting, from Eq. (1), and using the thin shear layer approximation,

$$C_{\epsilon 3}\chi = \left(\frac{C_{\epsilon 3}}{4C_\mu}\right)\left(\frac{G}{H}\right)^3 U'^2\left(C_\mu^{-1/2}\frac{V}{\eta}\right) \quad \text{round jet} \tag{8b}$$

$$C_{\epsilon 3}\chi = \left(\frac{C_{\epsilon 3}}{4C_\mu}\right)\left(\frac{G}{H}\right)^3 U'^2 U \quad \text{radial jet} \tag{8c}$$

so, from Eq. (2), a common formula is

$$C_{\epsilon 3}\chi = \bar{C}_{\epsilon 3}(G/H)^3 U'^2 F' \tag{8d}$$

It is evident from Eqs. (8b) and (8c) that the vortex stretching source of dissipation implied for the round jet has a radial jet counterpart. Moreover, the strain rate v/y becomes negative in the outer region of the round jet whereas the strain rate u/x is positive across the entire radial jet. For this reason, it may be expected that the hypothesized increase in dissipation by vortex stretching will be greater in the radial jet than in the round jet.

Results and Conclusions

Using the technique of Paullay et al.,[8] the far-field similarity equations were solved for a variety of free jet cases (see Table 1). The results support the contention that the Pope modification to the k-ϵ model affects the radial jet more than the round jet. The growth rate is reduced by 58% for the former and 28% for the latter. Unfortunately, round jet observations[3] agree with the modified model calculation while radial jet experiments[10,11] support the original model calculation. The plane jet calculations are known to match experiments[2,8] and are shown for comparison. It would appear that the round jet/plane jet anomaly has been exchanged for a round jet/radial jet anomaly.

These results indicate that radial far-field behavior can impose an additional constraint on turbulence models. The similarity form of the thin shear layer equations can aid in determining the consequences of model modification. Such a vortex stretching modification to the k-ϵ model fails when applied to the radial jet.

References

[1] Launder, B. E., Morse, A. P., Rodi, W., and Spalding, D. B., "The Prediction of Free Shear Flows—A Comparison of Six Turbulence Models," NASA SP-322, 1972.

[2] Launder, B. E. and Spalding, D. B., "The Numerical Computation of Turbulent Flows," *Computational Methods in Applied Mechanics and Engineering*, Vol. 3, 1974, pp. 269-289.

[3] Rodi, W., "The Prediction of Free Turbulent Boundary Layers by Use of a Two-Equation Model of Turbulence," Ph.D. Thesis, University of London, England, 1972.

[4] Pope, S. B., "An Explanation of the Turbulent Round-Jet/Plane-Jet Anomaly," *AIAA Journal*, Vol. 16, 1978, pp. 279-281.

[5] Raiszadeh, F. and Dwyer, H. A., "A Study with Sensitivity Analysis of the k-ϵ Turbulence Model Applied to Jet Flows," AIAA Paper 83-0285, Jan. 1983.

[6] McGuirk, J. J. and Rodi, W., "The Calculation of Three Dimensional Free Jets," *Turbulent Shear Flows I*, Springer-Verlag, Berlin, 1979, pp. 71-83.

[7] Bradshaw, P., "Complex Strain Fields," *The 1980-81 AFOSR-HTTM-Stanford Conference on Complex Turbulent Flows: Comparison of Computation and Experiment*, Vol. II, Stanford University, Stanford, Calif., 1982, pp. 700-712.

[8] Paullay, A. J., Melnik, R. E., Rubel, A., Rudman, S., and Siclari, M. J., "Similarity Solutions for Plane and Radial Jets Using a k-ϵ Turbulence Model," *Journal of Fluids Engineering, Transactions of the ASME*, Vol. 107, March 1985, pp. 79-85.

[9] Rubel, A. and Melnik, R. E., "Jet, Wake and Wall Jet Similarity Solutions Using a k-ϵ Turbulence Model," AIAA Paper 84-1523, June 1984.

[10] Tanaka, T. and Tanaka, E., "Experimental Study of a Radial Turbulent Jet," *Bulletin of the Japan Society of Mechanical Engineers*, Vol. 19, 1976, pp. 792-799.

[11] Witze, P. O. and Dwyer, H. A., "The Turbulent Radial Jet," *Journal of Fluid Mechanics*, Vol. 75, 1976, pp. 401-417.

Simplified Implicit Block-Bidiagonal Finite Difference Method for Solving the Navier-Stokes Equations

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Introduction

MUCH effort has been expended in recent times in developing a reliable and efficient method for solving the Navier-Stokes equations in two and three dimensions. MacCormack[1] introduced recently an implicit method based on his earlier explicit predictor-corrector scheme[2] that promised a significant increase in computational efficiency while retaining the simplicity of the explicit algorithm. The method was subsequently further developed and studied by several other investigators. Von Lavante and Thompkins[3] extended the method to curvilinear coordinate systems; Casier et al.[4] studied a class of schemes based on the bidiagonal solution technique and discussed several boundary condition procedures. Kordulla and MacCormack[5] applied a modified version of the aforementioned algorithm to transonic inviscid and viscous flow calculations about several airfoils. Although mostly encouraging results were reported, stability problems were encountered in cases with relatively strong shocks. Besides, it was felt that the computational efficiency of this method should still be improved. The present work, motivated by these results, studied the effects of simplifying the original implicit block-bidiagonal algorithm by introducing its spectral normal form. The resulting scheme is twice as fast as the original method and much more robust. On the other hand, it tends to overestimate the viscous effects. It was therefore decided to combine the two methods in a procedure in which approximate results were first obtained by the spectral normal form of the implicit MacCormack scheme (SNIMC) and, then, the full implicit MacCormack scheme (FIMC) was applied as post-processor. The resulting procedure was tested on several test cases with favorable results.

Received May 21, 1984; revision received July 30, 1984.

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